Cohomological field theory and related topics

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1 Cohomological Field Theory

Let V be a finite-dimensional $\mathbb{C}$-vector space equipped with a non-degenerate symmetric bilinear form η. A cohomological field theory on (V, η) consists of a collection $\Omega = \{\Omega_{g,n}\}_{2g-2+n>0}$ of elements

$$\Omega_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}) \otimes (V^*)^{\otimes n}$$

satisfying the following axioms.

i. (Symmetry) Every $\Omega_{g,n}$ is S_n-invariant. In other words, $\Omega_{g,n}$ is a collection of S_n equivariant maps:

$$\Omega_{g,n} : V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n})$$

ii. (Naturality) Considering the gluing maps

$$q : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}, \quad r : \overline{\mathcal{M}}_{h,|I|+1} \times \overline{\mathcal{M}}_{g-h,|J|+1} \to \overline{\mathcal{M}}_{g,n}, \quad I \sqcup J = \{1, \ldots, n\},$$

we have

$$q^*\Omega(v_1 \otimes \cdots \otimes v_n) = \Omega_{g-1,n+2}(v_1 \otimes \cdots \otimes v_n \otimes \Delta)$$

$$r^*\Omega(v_1 \otimes \cdots \otimes v_n) = \Omega_{h,|I|+1} \otimes \Omega_{g-h,|J|+1} \left(\bigotimes_{i=1}^{|I|} v_i \otimes \Delta \otimes \bigotimes_{j=1}^{|J|} v_{|I|+j} \right)$$

where Δ is the bivector in $V \otimes V$ dual to the metric $\eta \in V^* \otimes V^*$.

iii*. (Flat unit) Assume the vector space V comes with a distinguished element $1 \in V$. Consider the forgetful map

$$p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}.$$

Then

$$p^*\Omega_{g,n}(v_1 \otimes \cdots \otimes v_n) = \Omega_{g,n+1}(v_1 \otimes \cdots \otimes v_n \otimes 1),$$
$$\Omega_{0,3}(v_1 \otimes v_2 \otimes 1) = \eta(v_1, v_2).$$

The CohFT satisfying (i)(ii) but not (iii) is called degenerate CohFTs.

We can define a quantum product on (V, η) by

$$\eta(v_1 \star v_2, v_3) = \Omega_{0,3}(v_1, v_2, v_3).$$

A CohFT with unit (V, η, Ω) is semisimple if $(V, \star, 1)$ is a semisimple $\mathbb{C}$-algebra. It means we find an idempotent basis e_i such that

$$e_i \star e_j = \delta_{ij} e_i.$$

The partition function of a CohFT $\Omega = \{\Omega_{g,n}\}$ with respect to a basis $\{e_1, \ldots, e_D\}$ of V is defined by

$$Z_\Omega(\hbar, t_k^\alpha) = \exp \sum_{g \geq 0} \hbar^{g-1} F_g^\Omega(t_*^*)$$

$$F_g^\Omega(t_*^*) = \sum_{n, \vec{\alpha}, \vec{k}} \frac{1}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n}) \cdot \prod_{j=1}^n \psi_j^{k_j} t_{k_j}^{\alpha_j}$$

where $\alpha_i \in \{1, \ldots, D\}$ and $k_j \in \mathbb{N}$. We also define the correlator

$$\langle \tau_{k_1}(v_1), \ldots, \tau_{k_n}(v_n) \rangle_g^\Omega := \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(v_1 \otimes \cdots \otimes v_n) \cdot \prod_{j=1}^n \psi_j^{k_j}$$

Remark 1.1. *We can also use* $\sum_{g,n}\frac{\hbar^{2g-2}}{n!}F_{g,n}$*, or* $\sum_{g,n}\frac{\hbar^{2g-2+n}}{n!}F_{g,n}$ *in the defintion of partition function.*

Definition 1.2. *Given a CohFT* Ω*, the topological CohFT* Ω^{top} *is define as the degree 0 part of* Ω*:*

$$\Omega^{top}_{g,n}(v_1\otimes\cdots\otimes v_n)=\deg_0\Omega_{g,n}(v_1\otimes\cdots\otimes v_n)\in H^0(\overline{\mathcal{M}}_{g,n}).$$

A fundamental example of CohFT comes from the Gromov-Witten theory. Let X be a nonsingular projective variety.

$$V=H^*(X;\mathbb{C}),\quad \eta(v_1,v_2)=\int_X v_1\cup v_2.$$

$$\Omega^X_{g,n}(v_1\otimes\cdots\otimes v_n)=\sum_{\beta\in H_2(X;\mathbb{C})}q^\beta\Omega^X_{g,n,\beta}(v_1\otimes\cdots\otimes v_n)$$

$$\Omega^X_{g,n,\beta}(v_1\otimes\cdots\otimes v_n)=\pi_*\left([\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}\cap\prod_{i=1}^n \mathrm{ev}_i^*v_i\right)$$

where $\pi:\overline{\mathcal{M}}_{g,n}(X,\beta)\to\overline{\mathcal{M}}_{g,n}$ is the forgetful map. The Gromov-Witten CohFT partition function is about ancestor Gromov-Witten invariants

$$Z_X(\hbar,t_k^\alpha)=\exp\sum_{g\geq 0}\hbar^{g-1}\sum_{n,\beta,\vec{\alpha},\vec{k}}\frac{1}{n!}\langle\bar{\tau}_{k_1}(e_{\alpha_1})\dots\bar{\tau}_{k_n}(e_{\alpha_n})\rangle^X_{g,\beta}\prod_{j=1}^n t^{\alpha_j}_{k_j},$$

where $\bar{\psi}=\pi^*\psi$ and

$$\langle\bar{\tau}_{k_1}(e_{\alpha_1})\dots\bar{\tau}_{k_n}(e_{\alpha_n})\rangle^X_{g,\beta}:=\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}}\prod_{j=1}^n \mathrm{ev}_j^*(e_{\alpha_j})\bar{\psi}_j^{k_j}$$

One should notice that in many papers, the total ancestor potential has more information than $Z_X(\hbar,t_k^\alpha)$. We define the total ancestor potential

$$\mathcal{A}_t(\hbar;t_*^*):=\exp\sum_{g\geq 0}\hbar^{g-1}\bar{\mathcal{F}}_t^g,$$

where the genus g ancestor potential $\bar{\mathcal{F}}_t^g$ is defined by

$$\bar{\mathcal{F}}_t^g:=\sum_{n,m,\beta}\frac{q^\beta}{n!m!}\int_{[\overline{\mathcal{M}}_{g,n+m}(X,\beta)]^{\mathrm{vir}}}\wedge_{i=1}^n(\sum(\mathrm{ev}_i^*t_k)\bar{\psi}_i^k)\wedge_{i=n+1}^{n+m}\mathrm{ev}_i^*t.$$

$Z_X(\hbar,t_k^\alpha)$ is $\mathcal{A}_t(\hbar;t_*^*)$ at $t=0$. The total descendent potential of X is defined as

$$\mathcal{D}_X(\hbar;t_*^*):=\exp\sum_{g\geq 0}\hbar^{g-1}\mathcal{F}_X^g,$$

where the genus g descendent potential $\mathcal{F}_X^g$ is defined by

$$\mathcal{F}_X^g:=\sum_{n,\beta}\frac{q^\beta}{n!}\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}}\wedge_{i=1}^n(\sum(\mathrm{ev}_i^*t_k)\psi_i^k).$$

$\mathcal{D}_X$ has no dependence on the parameter t.

2 Givental's theory

Givental's group action gives a reconstruction method for semisimple cohomological field theories. It was first invented for Gromov-Witten theory with semisimple quantum cohomology. It is proven by the localization formula in the toric Fano case. Later, Teleman gave a complete classification for semisimple CohFTs to generalize this theory to all semisimple cases.

2.1 Givental's quantization formalism

Let $H, (.,.)$ be an N-dimensional vector space equipped with a non-degenerate symmetric bilinear form. Let $\mathcal{H}$ be the space of Laurent polynomials $\mathcal{H} = H((z^{-1}))$. Introduce a symplectic bilinear form in $\mathcal{H}$ by

$$\Omega(f,g) = \frac{1}{2\pi i}\oint (f(-z), g(z))dz$$

The $(\mathcal{H},\Omega)$ admits a polarization $\mathcal{H}_+ \oplus \mathcal{H}_-$

$$q \in \mathcal{H}_+ = H[z] \quad p \in \mathcal{H}_- = z^{-1}H[[z^{-1}]].$$

In application, $H = H^*(X;\mathbb{C}[[Q]])$ (V in CohFT), $(.,.)$ is the Poincare pairing (η in CohFT). Let q_k^α be coordinates of $\mathcal{H}_+$ and p_k^α be coordinates of $\mathcal{H}_-$ (called Darboux coordinate system on $(\mathcal{H},\Omega)$):

$$f(z) = \sum_{k\geq 0} q_k^\alpha \phi_\alpha z^k + \sum_{l\geq 0} p_l^\beta \eta^{\beta\epsilon}\phi_\epsilon(-z)^{-1-l} \in \mathcal{H}$$

Let A be an infinitesimal symplectomorphism $A : \mathcal{H} \to \mathcal{H}$. The quantized $\hat{A}$ is an at most second order differential operator acting on $\mathcal{H}_+$ defined as:

$$\hat{A} := \widehat{h_A(f)} = \frac{1}{2}\widehat{\Omega(Af,f)}, \quad f \in \mathcal{H}$$

$h_A(f)$ is a quadratic Hamiltonian in variables q, p. We set

$$\widehat{q_a q_b} = \hbar^{-1} q_a q_b, \quad \widehat{q_a p_b} = q_a \frac{\partial}{\partial q_b}, \quad \widehat{p_a p_b} = \hbar \frac{\partial^2}{\partial q_a \partial q_b}$$

In application, we use

$$\partial_{\alpha,k} := \frac{\partial}{\partial q_k^\alpha}.$$

Let $A = Bz^m$ where $B\phi_\beta = B^\alpha{}_\beta \phi_\alpha$. A direct computation shows that

- If $m < 0$, then

$$\hat{A} = \frac{1}{2\hbar}\sum_k (-1)^{k+m} B_{\alpha\beta} q_k^\beta q_{-1-k-m}^\alpha - \sum_k B^\alpha{}_\beta q_k^\beta \partial_{\alpha,k+m}$$

- If $m \geq 0$, then

$$\hat{A} = -\sum_k B^\alpha_\beta q^\beta_k \partial_{\alpha,k+m} + \frac{\hbar}{2}\sum_k (-1)^k B^{\alpha\beta}\partial_{\beta,k}\partial_{\alpha,m-1-k}$$

Remark 2.1. *In calculation, we need A is symplectic. i.e.* $A^*(-z) + A(z) = 0$ *and* $A^*_m = (-1)^{m+1} A_m$ *to cancel some (-1) sign.*

The quantization of a symplectomorphism $T = \exp A$ is defined as $\hat{T} := \exp \hat{A}$. The Birkhoff factorization can decompose a general symplectomorphism into a composition of uppertriangle S and lower triangle R: $T = S \circ R$.

Proposition 2.2. Consider a symplectomorphism of $\mathcal{H}$ of the form

$$S(z) = I + S_1/z + S_2/z^2 + \cdots \in \mathrm{End}(H^*(X))[[z^{-1}]].$$

Define a quadratic form W_S on $\mathcal{H}_+$ by the equation

$$W_S(\mathbf{q}) = \sum_{k,l\geq 0} (W_{kl}\mathbf{q}_k, \mathbf{q}_l),$$

where $\mathbf{q}_k = q^\alpha_k \phi_\alpha$ and W_{kl} is defined by

$$\sum_{k,l\geq 0} \frac{W_{kl}}{z^k w^l} = \frac{S^*(w)S(z) - I}{w^{-1} + z^{-1}}$$

Then the quantization of S^{-1} acts on the Fock space by

$$(\widehat{S^{-1}}\mathcal{G})(\mathbf{q}) = \exp\left(\frac{W_S(\mathbf{q})}{2\hbar}\right)\mathcal{G}([S\mathbf{q}]_+)$$

for any function $\mathcal{G}$ of $\mathbf{q} \in \mathcal{H}_+$. Here $[S\mathbf{q}]_+$ denotes the truncation of $S(z)\mathbf{q}$ to a power series in z.

Proof. See [2]. □

A basic example comes from the fundamental solution

$$(S_\tau(z)u, v) = (u, v) + \ll \frac{u}{z-\psi}, v \gg_{0,2} (\tau) \quad \forall u, v \in H^*(X;\Lambda)$$

where

$$\ll a_1, a_2, \ldots, a_m \gg_{g,m} (\tau) := \sum_{n,\beta} \frac{Q^\beta}{n!}\langle a_1, a_2, \ldots, a_m, \tau, \ldots, \tau\rangle_{g,n+m,\beta}$$

In our notation,

$$S_{\alpha\beta}(z) = (S_\tau(z)\phi_\beta, \phi_\alpha) = \eta_{\alpha\beta} + \ll \frac{\phi_\beta}{z-\psi}, \phi_\alpha \gg_{0,2} (\tau)$$

$$S^\alpha_\beta = \delta^\alpha_\beta + \ll \frac{\phi_\beta}{z-\psi}, \phi^\alpha \gg_{0,2} (\tau)$$

By WDVV equation, e.g [3, Prop 1.4.1] we know

Proposition 2.3.

$$S_{\mu\alpha}(w)\eta^{\mu\nu}S_{\nu\beta}(z) = (z+w) \ll \frac{\phi_\alpha}{w-\psi}, \frac{\phi_\beta}{z-\psi} \gg_{0,2} (\tau) + \eta_{\alpha\beta}$$

Then

$$\begin{aligned}[S^*(w)S(z)]^\mu_{\ \nu} &= [S^*(w)]^\mu_{\ \gamma}[S(z)]^\gamma_{\ \nu} = S(w)_\gamma^{\ \mu}S(z)^\gamma_{\ \nu} \\ &= (z+w) \ll \frac{\phi^\mu}{w-\psi}, \frac{\phi_\nu}{z-\psi} \gg_{0,2} (\tau) + \delta^\mu_\nu\end{aligned}$$

As a corollary, $S_\tau(z)$ is a symplectomorphism: $S^*(-z)S(z) = I$.

$$\frac{S^*(w)S(z) - I}{w^{-1}+z^{-1}} = \sum_{a,b\geq 0} w^{-a}z^{-b} \ll \phi^\mu\psi^a, \phi_\nu\psi^b \gg_{0,2} (\tau)$$

$$(W_{kl})^\mu_{\ \nu} = \ll \phi^\mu\psi^l, \phi_\nu\psi^k \gg_{0,2} (\tau)$$

The quadratic form is

$$\begin{aligned}W_S(\mathbf{q}) &= \sum_{k,l\geq 0}(W_{kl}q_k, q_l) = \sum_{k,l}(W_{kl}q_k^\alpha\phi_\alpha, q_l^\beta\phi_\beta) \\ &= \sum_{k,l}((W_{kl})^\gamma_{\ \alpha}q_k^\alpha\phi_\gamma, q_l^\beta\phi_\beta) = \sum_{k,l} \ll \phi^\gamma\psi^l, \phi_\alpha\psi^k \gg_{0,2} q_k^\alpha q_l^\beta\eta_{\gamma\beta} \\ &= \sum_{k,l} \ll \phi_\alpha\psi^k, \phi_\beta\psi^l \gg_{0,2} q_k^\alpha q_l^\beta = \ll \mathbf{q}, \mathbf{q} \gg_{0,2} (\tau)\end{aligned}$$

In this case,

$$(\widehat{S^{-1}}\mathcal{G})(\mathbf{q}) = e^{(1/2\hbar)\ll\mathbf{q},\mathbf{q}\gg_{0,2}(\tau)}\mathcal{G}([S\mathbf{q}]_+)$$

Now, let's do the explicit calculation of $[S\mathbf{q}]_+$.

$$S(z)\mathbf{q} = \sum_{k,l} z^{-k+l}S_k q_l^\beta\phi_\beta = \sum_{k,l} z^{-k+l}(S_k)^\alpha_{\ \beta}q_l^\beta\phi_\alpha$$

$$[S\mathbf{q}]_+ = \sum_{m\geq 0} z^m\left(\sum_{l\geq m}(S_{l-m})^\alpha_{\ \beta}q_l^\beta\phi_\alpha\right)$$

In this case, $S_0 = I$; for $k \geq 1$,

$$(S_k)^\alpha_{\ \beta} = \ll \phi^\alpha, \phi_\beta\psi^{k-1} \gg_{0,2} (\tau).$$

There is a similar theorem for lower-triangular symplectormorphism.

Proposition 2.4. Let R be a lower-triangular symplectomorphism of $\mathcal{H}$ of the form

$$R(z) = I + R_1 z + R_2 z^2 + \dots.$$

Define a quadratic form V_R on $\mathcal{H}_-$ by the equation

$$V_R(\mathbf{p}_0(-z)^{-1} + \mathbf{p}_1(-z)^{-2} + \mathbf{p}_2(-z)^{-3} + \dots) = \sum_{k,l\geq 0}(\mathbf{p}_k, V_{kl}\mathbf{p}_l),$$

where $\mathbf{p}_k = p_k^\alpha \phi^\alpha$ and V_{kl} is defined by

$$\sum_{k,l\geq 0}(-1)^{k+l}V_{kl}w^k z^l = \frac{R^*(w)R(z)-I}{z+w}.$$

Then the quantization of R acts on the Fock space by

$$(\widehat{R}\mathcal{G})(\mathbf{q}) = \left[\exp\left(\frac{\hbar V_R(\partial_\mathbf{q})}{2}\right)\mathcal{G}\right](R^{-1}\mathbf{q})$$

where $V_R(\partial_\mathbf{q})$ is the differential operator obtained from $V_R(\mathbf{p})$ by replacing p_k by $\frac{\partial}{\partial q_k}$.

To use this proposition, we compute:

$$\begin{aligned} V_R(\mathbf{p}) &= \sum_{k,l}(p_k^\alpha \phi^\alpha, V_{kl}p_l^\beta \eta^{\beta\epsilon}\phi_\epsilon) = \sum_{k,l}(p_k^\alpha\phi^\alpha, (V_{kl})^\tau{}_\epsilon \phi_\tau p_l^\beta \eta^{\beta\epsilon}) \\ &= \sum_{k,l} p_k^\alpha (V_{kl})^\alpha_\epsilon p_l^\beta \eta^{\beta\epsilon} = \sum_{k,l} p_k^\alpha p_l^\beta [V_{kl}\eta^{-1}]^\alpha_\beta \end{aligned}$$

From the expression of V_R, it is natural to define

$$V_{kl}^{\alpha\beta} := [V_{kl}\eta^{-1}]^\alpha_\beta = (V_{kl})^\alpha_\gamma \eta^{\gamma\beta}.$$

$$V_R(\partial_\mathbf{q}) = \sum_{k,l} V_{kl}^{\alpha\beta}\frac{\partial}{\partial q_k^\alpha}\frac{\partial}{\partial q_l^\beta}$$

In terms of symplecity, $R^*(z) = R^{-1}(-z)$,

$$\sum_{k,l\geq 0}(-1)^{k+l}V_{kl}w^k z^l = \frac{R^*(w)R(z)-I}{z+w} = \frac{R^{-1}(-w)[R^{-1}]^*(-z)-I}{z+w}$$

$$\sum_{k,l\geq 0}V_{kl}w^k z^l = \frac{I - R^{-1}(w)(R^{-1})^*(z)}{z+w}$$

$$\sum_{k,l\geq 0}V_{kl}\eta^{-1}w^k z^l = \frac{\eta^{-1} - R^{-1}(w)\eta^{-1}(R^{-1})^T(z)}{z+w}$$

$$V_{kl}^{\alpha\beta} = [w^k z^l]\frac{\eta^{\alpha\beta} - (R^{-1})^\alpha_\gamma(w)\eta^{\gamma\epsilon}(R^{-1})^\beta_\epsilon(z)}{z+w}$$

Remark 2.5. *Notice that [19] use different notation by replacing R with R^{-1}. This is explained in [19, annotation 11, page 11]*

The quantization is a fundamental constraint in Gromov-Witten invariants.

Theorem 2.6. The string equation can be described as

$$\widehat{\left(\frac{1}{z}\right)}\mathcal{D}_X = 0$$

The following theorem builds a relation of descendent and ancestor potential

Theorem 2.7 ([4, 14]). Let

$$F^1(\tau) = \mathcal{F}_X^1(\mathbf{t})|_{t_0=\tau, t_1=t_2=\cdots=0}$$

denote the genus 1 nondescendent Gromov-Witten potential of X. Then

$$\mathcal{D}_X = e^{F^1(t)}\widehat{S_\tau^{-1}}\mathcal{A}_\tau$$

Proof. See [3, Theorem 1.5.1] □

It is worth to remark that the RHS does not depend on τ.

2.2 The Frobenius manifold and R-matrix

2.2.1 Frobenius manifold

The Frobenius manifold consists of $(M, \eta, A, \mathbf{1})$:

- M is a complex manifold of dimension n.
- η is a holomorphic, symmetric, nondegenerate quadratic form on the complex tangent bundle TM. It plays the row of flat holomorphic metric on M.
- A is a holomorphic symmetric tensor

  $$A : TM \otimes TM \otimes TM \to \mathcal{O}_M.$$

  It plays the role of $\Omega_{0,3}$ in CohFT.
- $\mathbf{1}$ is a holomorphic vector field on M.

Locally, M is covered by open subsets $U = \mathrm{Spec}\mathbb{C}[t^1, \ldots, t^n]$ such that there exists η-flat vector fields

$$\partial_1 = \frac{\partial}{\partial t^1}, \ldots, \partial_n = \frac{\partial}{\partial t^n} \in \Gamma(U, TM),$$

and a holomorphic function

$$F(t^1, \ldots, t^n) \equiv F(t) \in \Gamma(U, \mathcal{O}_M).$$

such that

- ► $\partial_\alpha\partial_\beta\partial_\lambda F(t)\eta^{\lambda\mu}\partial_\mu\partial_\gamma\partial_\delta F(t) = \partial_\delta\partial_\beta\partial_\lambda F(t)\eta^{\lambda\mu}\partial_\mu\partial_\gamma\partial_\alpha F(t)$

- $\partial_\alpha\partial_\beta\partial_1 F(t) \equiv \eta_{\alpha\beta} = \eta(\partial_\alpha, \partial_\beta)$
- $\partial_\alpha\partial_\beta\partial_\gamma F(t) = \eta(\partial_\alpha \star \partial_\beta, \partial_\gamma)$
- $c^\gamma_{\alpha\beta}(t) := \eta^{\gamma\epsilon}\partial_\epsilon\partial_\alpha\partial_\beta F(t)$ and $\partial_\alpha \star \partial_\beta = c^\gamma_{\alpha\beta}(t)\partial_\gamma$

In Dubrovin's original papers, we require the quasihomogenerity condition on the Frobenius manifold. That is there exists an Euler vector field

$$E = \sum_{\alpha=1}^{n} [(1-q_\alpha)t^\alpha + r_\alpha]\partial_\alpha,$$

such that

$$\mathcal{L}_E F(t) := \sum_{\alpha=1}^{n} [(1-q_\alpha)t^\alpha + r_\alpha]\partial_\alpha F(t) = (3-d)F(t) + \frac{1}{2}A_{\alpha\beta}t^\alpha t^\beta + B_\alpha t^\alpha + C$$

for some constant $A_{\alpha\beta}, B_\alpha, C$. A Frobenius manifold with an Euler vector field is called conformal. The quasihomogenerity can be translated into:

$$\begin{aligned} \nabla_\gamma(\nabla_\beta E^\alpha) &= 0 \\ (\mathcal{L}_E c)^\gamma_{\alpha\beta} &= c^\gamma_{\alpha\beta} \\ \mathcal{L}_E \mathbf{1} &= -\mathbf{1} \\ (\mathcal{L}_E \eta)_{\alpha\beta} &= (2-d)\eta_{\alpha\beta} \end{aligned}$$

In our application, X is the toric Fano variety. The Frobenius manifold is $(H = H^*(X;\mathbb{C}[[Q]]), \eta, \Omega^X_{0,3}, \mathbf{1})$. A general element in $H^*(X)$ is reprecented by $t^0\mathbf{1} + \sum_{i=1}^r t^i\phi_i + \sum_{i=r+1}^N t^i\phi_i$. $H^2(X) = \text{span}\{\phi_i\}_{i=1}^r$. The Euler vector field is

$$E = t^0\frac{\partial}{\partial t^0} + \sum_{i=1}^{r}\rho^i\frac{\partial}{\partial t^i} + \sum_{i=r+1}^{N}(1 - \frac{1}{2}\deg\phi_i)t^i\frac{\partial}{\partial t^i}$$

where $c_1(X) = \rho^1\phi_1 + \cdots + \rho^r\phi_r$. The grading operator $\mu : H \to H$ is defined as

$$\mu(\phi_i) = (\frac{1}{2}\deg\phi_i - \frac{1}{2}\dim_{\mathbb{C}} X)\phi_i.$$

I summarize the notations from different materials, including [5–7, 14, 17], in Figure 1.

The $(*)$ in Figure 1 is somehow subtle. We should understand $(*)$ as an expression in local coordinates. The matrix $\mathcal{U}$ is from the quantum product:

$$E * \partial_\alpha = E^\gamma c^\beta_{\gamma\alpha}\partial_\beta; \quad \mathcal{U}^\beta_\alpha := E^\gamma c^\beta_{\gamma\alpha}$$

$\mu = diag(\mu_0, \mu_1, \ldots, \mu_N)$. The element $\xi = \xi_\beta dt^\beta$ is a flat section of cotangent bundle w.r.t $\tilde{\nabla}$. Transform the Dubrovin connection into cotangent bundle:

$$\tilde{\nabla}_\alpha\xi_\beta = \partial_\alpha\xi_\beta - zc^\gamma_{\alpha\beta}\xi_\gamma,$$

	Dubrovin	Givental(i)	Givental(ii)
QDE	$\tilde{\nabla}_u v = \nabla_u v + zu * v$ $\tilde{\nabla}_{\frac{\partial}{\partial z}} v = \partial_z v + E * v - \frac{1}{z}\mu v$ $\partial_z \xi_\alpha = \mathcal{U}^\beta{}_\alpha \xi_\beta - \frac{1}{z}\mu_\alpha \xi_\alpha$ $(*)$	$\nabla^z_{\frac{\partial}{\partial t^i}} = \frac{\partial}{\partial t^i} - \frac{1}{z}(\phi_i *)$ $\nabla^z_{z\frac{\partial}{\partial z}} = z\frac{\partial}{\partial z} + \frac{1}{z}(E*) + \mu$ $(z\partial_z + E)\xi_\alpha = \mu_\alpha \xi_\alpha$ $(\star)$	$\nabla^z_{\frac{\partial}{\partial t^i}} = \frac{\partial}{\partial t^i} + \frac{1}{z}(\phi_i *)$ $\nabla^z_{z\frac{\partial}{\partial z}} = z\frac{\partial}{\partial z} - \frac{1}{z}(E*) + \mu$
$\dim_{\mathbb{C}} X$	d	D	
Hodge grading	$\mu = \frac{2-d}{2} - \nabla E$ $\mu_\alpha = q_\alpha - \frac{d}{2}$	$\phi_\alpha \in H^{2q_\alpha}(X)$ $d_\alpha = q_\alpha$	
	$\theta_{\alpha,k} = h_{\alpha,k}$ $\Omega_{\alpha,k;\beta,l}$	$\ll \tau_k(\phi_\alpha), 1 \gg^X_{0,2} (t)$ $\ll \tau_k(\phi_\alpha), \tau_l(\phi_\beta) \gg^X_{0,2} (t)$	

Figure 1: Dictionary of notations

$$\tilde{\nabla}_{\frac{\partial}{\partial z}} \xi_\beta = \partial_z \xi_\beta - \mathcal{U}^\alpha_\beta \xi_\alpha + \frac{1}{z}\mu_\beta \xi_\beta.$$

Set we get the equation $(*)$. To see $(\star)$, we can

$$\begin{aligned}\partial_z \xi_\alpha &= \mathcal{U}^\beta{}_\alpha \xi_\beta - \frac{1}{z}\mu_\alpha \xi_\alpha \\ &= E^\gamma c^\beta_{\gamma\alpha} \xi_\beta - \frac{1}{z}\mu_\alpha \xi_\alpha \\ &= \frac{1}{z}(E^\gamma \partial_\gamma \xi_\alpha - \mu_\alpha \xi_\alpha)\end{aligned}$$

Take $w = -z^{-1}$, $\partial_z = w^2\partial_w$, $z\partial_z = -w\partial_w$, we get

$$(w\partial_w + \partial_E)\xi_\alpha = \mu_\alpha \xi_\alpha.$$

If we do a linear change of coordinates $\xi^\# = \eta^{-1}\xi^T$ ($\xi^\alpha = \eta^{\alpha\beta}\xi_\beta$). Apply the η-symmetry: $\mathcal{U}^T\eta = \eta\mathcal{U}$, $\mu\eta + \eta\mu = 0$,

$$\eta^{\gamma\alpha}\mathcal{U}^\beta_\alpha = \mathcal{U}^\gamma{}_\alpha \eta^{\alpha\beta}; \quad \eta^{\alpha\beta}\mu_\beta + \mu_\alpha \eta^{\alpha\beta} = 0$$

we can also see an expression in [7, Equation 2.41]:

$$\partial_z \xi^\gamma = \mathcal{U}^\gamma_\beta \xi^\beta + \frac{1}{z}\mu_\gamma \xi^\gamma; \quad \partial_z \xi^\# = \left(\mathcal{U} + \frac{1}{z}\mu\right)\xi^\#$$

We should notice that this form is a local expression of QDE, which has a different sign compared with $\tilde{\nabla}_{\frac{\partial}{\partial z}} v$.

2.2.2 canonical coordinates

We assume the Frobenius manifold is semisimple. The Greek indices $\{t^\alpha\}$, $\{\partial_\alpha\}$ is for flat coordinates. Let $u^i = u^i(t^\alpha)$ be canonical coordinates and $\epsilon_i = \frac{\partial}{\partial u^i}$ be coordinate vector fields.

$$\epsilon_i \star \epsilon_j = \delta_{ij}\epsilon_i; \quad \Delta_i := \langle \epsilon_i, \epsilon_i \rangle^{-1}.$$

The normalized vector fields are

$$\tilde{\epsilon}_i = \sqrt{\Delta_i}\epsilon_i$$

$$\langle \tilde{\epsilon}_i, \tilde{\epsilon}_j \rangle = \delta_{ij}$$

The transition matrix Ψ is defined by

$$\partial_\mu = \Psi^i{}_\mu \tilde{\epsilon}_i; \quad \Delta_i^{-1/2} du^i = \Psi^i{}_\beta dt^\beta$$

$$\Psi^i{}_\mu = \langle \tilde{\epsilon}_i, \partial_\mu \rangle = \Delta_i^{-1/2} \frac{\partial u^i}{\partial t^\mu}$$

In particular,

$$\sum_i \Psi^i{}_\alpha \Psi^i{}_\beta = \eta_{\alpha\beta}; \quad \Psi^i{}_\mu \eta^{\mu\nu} \Psi^j{}_\nu = \delta_{ij}$$

There is a tablet to show difference of Ψ in different references.

Ψ	[17, 19], this note	[13]	[8, 14, 15]
	row up; column down	row up; column down	row down; column up
	$\partial_\mu = \Psi^i{}_\mu \tilde{\epsilon}_i$	$\Psi : \mathbb{C}^N \to H$ $e_i \mapsto \sqrt{\Delta_i}\partial_{u^i}$	$\Psi : H \to \mathbb{C}^N$ $\sqrt{\Delta_i}\partial_{u^i} \mapsto e_i$
	$\Psi^i{}_\mu = \Delta_i^{-1/2} \frac{\partial u^i}{\partial t^\mu}$	$\Psi^\alpha_i = \sqrt{\Delta_i} \frac{\partial t^\alpha}{\partial u^i}$ $\left(\Psi^{-1}\right)^i{}_\mu = \Delta_i^{-1/2} \frac{\partial u^i}{\partial t^\mu}$	$\Psi_\mu{}^i = \Delta_i^{-1/2} \frac{\partial u^i}{\partial t^\mu}$ $(\Psi^{-1})_i{}^\alpha = \sqrt{\Delta_i} \frac{\partial t^\alpha}{\partial u^i}$
	$\Psi^T \Psi = \eta$ $\Psi \eta^{-1} \Psi^T = I$	$\Psi \Psi^T = \eta^{-1}$ $\Psi^T \eta \Psi = I$ $\Psi^\alpha_i (\partial_\alpha, \partial_\beta) \Psi^\beta_j = \delta_{i,j}$	$\Psi \Psi^T = \eta$ $\Psi^T \eta^{-1} \Psi = I$ $\Psi_\mu{}^i \eta^{\mu\nu} \Psi_\nu{}^j = \delta_{ij}$

2.2.3 fundamental solution

A matrix in normalized canonical coordinates $\tilde{S}^i{}_j$ is a fundamental solution if the vector field $X_j = \tilde{S}^i{}_j \tilde{\epsilon}_i$ satisfies $\nabla^z_\alpha X_j = 0$ for all α, j (not include z direction).

Theorem 2.8 (the fundamental solution)**.** The differential equation in canonical coordinates for ∇^z-flat fields has the following properties:

- Formal fundamental solutions $\tilde{S}^i{}_j$ is of the form
$$\tilde{S}(z,u) = R(z,u)e^{\mathbf{u}/z},$$
where $R(z,u) = I + zR_1(u) + z^2R_2(u) + \dots$.
- $R(z,u)$ satisfies the unitary condition (R^t is transpose, it is true because $\eta_{ij} = \delta_{ij}$ in normalized canonical coordinates)
$$R(z,u)R^t(-z,u) = I$$
- $R(z,u)$ is unique up to a constant matrix (w.r.t u)
$$\exp\left(\sum_{k\geq 1} \mathbf{a}_{2k-1} z^{2k-1}\right)$$
- There is a unique R-matrix such that $(\star)$ holds.

The $\xi^\#$ version of $(\star)$ is here. In general coordinates, μ is not a diagonal matrix. We set $\mu(\partial_\beta) = M^\epsilon_\beta \partial_\epsilon$ and
$$\eta^{\alpha\beta}M^\beta_\gamma + M^\alpha_\beta\eta^{\beta\gamma} = 0$$
In $(*)$, we get
$$\partial_z\xi_\beta = \mathcal{U}^\alpha_\beta\xi_\alpha - \frac{1}{z}M^\epsilon_\beta\xi_\epsilon.$$
$$\partial_z\xi^\alpha = \mathcal{U}^\alpha_\beta\xi^\beta + \frac{1}{z}M^\epsilon_\alpha\xi^\epsilon.$$
Translate it into the Givental side:
$$(w\partial_w + \partial_E)\xi^\alpha = -M^\beta_\alpha\xi^\beta.$$
Hence quasihomogenerity condition for R-matrix is
$$(z\partial_z + \partial_E)R^i{}_j = -M^k_i R^k{}_j.$$

The fundamental solution $S^\gamma{}_j$ in flat coordinates is of the form
$$S^\gamma{}_j = (\Psi^{-1})^\gamma{}_k \tilde{S}^k{}_j$$
such that $Y_j = S^\gamma{}_j\partial_\gamma$ satisfies $\nabla^z_\alpha Y_j = 0$ for all α, j. Since we use Ψ^{-1} in the expansion of fundamental solutions, the statements about Ψ are the inverse of [13] (see [13, Page 10 annotation 7]), but they coincide with [17]. Moreover, although [8, Prop 2.3] write $\hat{\Psi}\hat{R}$, it plays the role as $\widehat{\Psi^{-1}}$ (see [8, Prop 5.1]).

2.2.4 R-matrix

Under the $\{\partial_\mu\}$ basis, the Levi-Civita connection is $\nabla = d$. i.e.

$$\Gamma^\gamma_{\alpha\beta} = (\Gamma_\alpha)^\gamma_{\ \beta} = 0$$

Lemma 2.9. In the $\{\tilde{\epsilon}_i\}$ basis,

$$\nabla = d + \Psi d\Psi^{-1},$$

$$\Gamma^k_{ij} = [\Psi\tilde{\epsilon}_i\Psi^{-1}]^k_{\ j} = \Psi^k_{\ \beta}\tilde{\epsilon}_i(\Psi^{-1})^\beta_{\ j}$$

Proof.

$$\nabla_{\tilde{\epsilon}_i}\tilde{\epsilon}_j = \nabla_{(\Psi^{-1})^\alpha_{\ i}\partial_\alpha}(\Psi^{-1})^\beta_{\ j}\partial_\beta = (\Psi^{-1})^\alpha_{\ i}\partial_\alpha(\Psi^{-1})^\beta_{\ j}\partial_\beta = (\Psi^{-1})^\alpha_{\ i}\partial_\alpha(\Psi^{-1})^\beta_{\ j}\Psi^k_{\ \beta}\tilde{\epsilon}_k$$

$$(\Gamma_i)^k_{\ j} = (\Psi^{-1})^\alpha_{\ i}\partial_\alpha(\Psi^{-1})^\beta_{\ j}\Psi^k_{\ \beta} = (\Psi^{-1})^\alpha_{\ i}\Psi^l_{\ \alpha}\tilde{\epsilon}_l(\Psi^{-1})^\beta_{\ j}\Psi^k_{\ \beta} = \Psi^k_{\ \beta}\tilde{\epsilon}_i(\Psi^{-1})^\beta_{\ j}$$ □

Remark 2.10. *In Givental's "row down column up" notation, we are working in the dual basis* $\{\tilde{\epsilon}^i = \frac{1}{\sqrt{\Delta_i}}du^i\}$*,*

$$\tilde{\epsilon}^i = dt^\beta\Psi_\beta^{\ i}; \quad \nabla_{\tilde{\epsilon}_i}\tilde{\epsilon}^j = (\Gamma^*_i)_k^{\ j}\tilde{\epsilon}^k = (\Gamma^*)^j_{ik}\tilde{\epsilon}^k.$$

Then in the dual basis $\{\tilde{\epsilon}^i\}$*,* $\nabla = d + \Psi^{-1}d\Psi$*,*

$$(\Gamma^*)^j_{ik} = (\Psi^{-1})_k^{\ \beta}\tilde{\epsilon}_i\Psi_\beta^{\ j}.$$

Lemma 2.11. In the $\{\tilde{\epsilon}_i\}$ basis, the quantum connection is written as

$$\nabla^z = \nabla - \frac{1}{z}d\mathbf{u}$$

where $\mathbf{u}$ is the diagonal matrix: $\mathbf{u} = \mathrm{Diag}(u^1, \ldots, u^n)$

Proof. $u^i = \sqrt{\Delta^i}\tilde{u}^i$

$$[d\mathbf{u}]_i(\tilde{\epsilon}_j) = [\frac{\partial u^j}{\partial\tilde{u}^i}]\tilde{\epsilon}_j = \sqrt{\Delta^i}\delta_{ij}\tilde{\epsilon}_j$$

In comparison,

$$\tilde{\epsilon}_i \star \tilde{\epsilon}_j = \sqrt{\Delta^i}\tilde{\epsilon}_i\delta_{ij}$$ □

By the flatness of ∇^z,

Lemma 2.12. $\Psi d\Psi^{-1} \wedge d\mathbf{u} + d\mathbf{u} \wedge \Psi d\Psi^{-1} = 0$

In the normalized canonical basis,

$$(d + \Psi d\Psi^{-1} - \frac{1}{z}d\mathbf{u})(Re^{\mathbf{u}/z}) = 0$$

$$dR \cdot e^{\mathbf{u}/z} + z^{-1}Re^{\mathbf{u}/z}d\mathbf{u} + \Psi d\Psi^{-1} \cdot Re^{\mathbf{u}/z} - \frac{1}{z}d\mathbf{u} \cdot Re^{\mathbf{u}/z} = 0$$

$$z(d + \Psi d\Psi^{-1})R = [d\mathbf{u}, R]$$

Expand the R-matrix in z-powers, we have

$$\Psi d\Psi^{-1} = [d\mathbf{u}, R_1], \quad (d + \Psi d\Psi^{-1})R_k = [d\mathbf{u}, R_{k+1}]$$

Remark 2.13. *In Givental's notation, Ψ is changed to be Ψ^{-1}.*

Remark 2.14. *The coefficient $(R_1)^i{}_j$ is the rotation coefficient γ_{ij} in [6, 7]. I give a proof in Appendix B.*

Theorem 2.15. The genus 1 non-descendent Gromov-Witten potential $F^1(\tau)$ is given by

$$F^1(\tau) = \frac{1}{48}\sum_i \ln \Delta_i(\tau) + C(\tau).$$

where

$$C(\tau) := \frac{1}{2}\int^{\tau} \sum_i (R_1)^i{}_i(u)du^i$$

The ancestor potential is defined by the formula

$$\mathcal{A}_\tau(\mathbf{q}_*^\alpha) = \widehat{\Psi^{-1}}\widehat{R}_\tau \widehat{e^{\mathbf{u}/z}} \prod_{i=1}^{N} \tau(\mathbf{q}_*^i)\Delta_i^{-1/48}(\tau)$$

Here, we use the dilaton shift: $\mathbf{q}(z) = \mathbf{t}(z) - \mathbf{1}z$, where $\mathbf{1}$ is the unit in $H^*(X)$. $(\mathbf{q}^1, \dots, \mathbf{q}^N)^T = \Psi(\mathbf{q}^\alpha)$ is the expression in normalized canonical coordinates. $\widehat{\Psi^{-1}}$ is the operator identifying the Fock space with its coordinate version $\mathcal{G}(\mathbf{q}^i) \mapsto \mathcal{G}(\mathbf{q}^\alpha)$ via $\mathbf{q}^i = \Psi^i{}_\alpha q^\alpha$. The tau function $\tau(\mathbf{t}_*)$, $\mathbf{t}_*(z) = t_0 + t_1 z + t_2 z^2 + \dots$ is the partition function of trivial CohFT:

$$\mathcal{F}_{\mathrm{pt}}^g(\mathbf{t}) = \sum_n \frac{1}{n!}\int_{\overline{\mathcal{M}}_{g,n}} \mathbf{t}(\psi) \wedge \cdots \wedge \mathbf{t}(\psi)$$

$$\tau(\hbar; \mathbf{t}) = \exp\left(\sum_{g=0}^{\infty} \hbar^{g-1}\mathcal{F}_{\mathrm{pt}}^g(\mathbf{t})\right)$$

Theorem 2.16. The total descendent potential of a semisimple Frobenius manifold is

$$\mathcal{D}(\hbar; \mathbf{t}) = e^{C(\tau)}\widehat{S_\tau^{-1}}\widehat{\Psi^{-1}}\widehat{R}_\tau \widehat{e^{\mathbf{u}/z}} \prod_{i=1}^{N} \tau(\mathbf{q}_*^i)$$

In the following paragraph, I try to explain this theorem more precisely.

- Because of the string equation, $e^{\widehat{\mathbf{u}/z}}$ does not really change $\mathcal{T} = \prod_{i=1}^{N} \tau(\mathbf{q}^i_*)$.

$$e^{\widehat{u^i/z}}\tau(\mathbf{q}^i) = \left(I + \sum_{k\geq 1} \frac{(\widehat{u^i/z})^k}{k!}\right)\tau(\mathbf{q}^i) = \tau(\mathbf{q}^i)$$

- By proposition 2.4, $\widehat{R}$-action is

$$\widehat{R}\mathcal{T}(\mathbf{q}^i) = [e^{\frac{\hbar}{2}\sum_{k,l\geq 0}\sum_{i,j} V^{ij}_{kl}\frac{\partial}{\partial q^i_k}\frac{\partial}{\partial q^j_l}} \prod_{m=1}^{N} \tau(\mathbf{q}^m)](R^{-1}\mathbf{q})$$

- Ψ^{-1} action is

$$[\widehat{\Psi^{-1}}\widehat{R}_\tau e^{\widehat{\mathbf{u}/z}}\mathcal{T}](\mathbf{q}^\alpha) = [e^{\frac{\hbar}{2}\sum_{k,l\geq 0}\sum_{i,j} V^{ij}_{kl}\frac{\partial}{\partial q^i_k}\frac{\partial}{\partial q^j_l}} \prod_{m=1}^{N} \tau(\mathbf{q}^m)](R^{-1}\Psi\mathbf{q})$$

- $\widehat{S_\tau^{-1}}$ action (this S_τ is in flat basis) is

$$[\widehat{S_\tau^{-1}}\widehat{\Psi^{-1}}\widehat{R}_\tau e^{\widehat{\mathbf{u}/z}}\mathcal{T}](\mathbf{q}^\alpha) = \Bigg(\exp(\frac{W_S(\mathbf{q})}{2\hbar})$$
$$\exp(\frac{\hbar}{2}\sum_{k,l\geq 0}\sum_{i,j} V^{ij}_{kl}\frac{\partial}{\partial q^i_k}\frac{\partial}{\partial q^j_l})\prod_{m=1}^{N}\tau(\mathbf{q}^m)\Bigg)(R^{-1}\Psi[S\mathbf{q}]_+)$$

- Consider $C(u)$, we get

$$\mathcal{D}(\hbar;\mathbf{q}^\alpha) = \Bigg(\exp(\frac{1}{2}\int^\tau \sum_i (R_1)^i{}_i(u)du^i)\exp(\frac{W_S(\mathbf{q})}{2\hbar})$$
$$\exp(\frac{\hbar}{2}\sum_{k,l\geq 0}\sum_{i,j} V^{ij}_{kl}\frac{\partial}{\partial q^i_k}\frac{\partial}{\partial q^j_l})\prod_{m=1}^{N}\tau(\mathbf{q}^m)\Bigg)(R^{-1}\Psi[S\mathbf{q}]_+)$$

2.3 Reconstruction of CohFTs

This section aims to understand Theorem 2.16 in terms of reconstruction of semisimple CohFT and explain the graph sum formula in this reconstruction.

We should realize that the expression of ancestor potential is important. It is the potential of Gromov-Witten CohFT. The tau function $\prod_{j=1}^{N}\tau(\hbar;\mathbf{q}^i)$ can be viewed as the potential of topological CohFTs Ω^{top}. The Ψ action is just a change of coordinates, this action can be absorbed by the R-matrix. The $e^{\widehat{\mathbf{u}/z}}$ plays no role in quantization. Thus we see the R matrix is the key point in the reconstruction of CohFTs.

Recall the definition of topological CohFT, also called 2D TQFT, $\Omega^{\text{top}}_{g,n}$ in Definition 1.2. The cohomology maps q^*, r^*, p^* are trivial because they are all restricted in $H^0(\cdot)$. Then $\Omega^{\text{top}}_{g,n}$ is determined by $\Omega_{0,3}$.

Lemma 2.17. In terms of normalized canonical basis $\{\tilde{\epsilon}_i\}$,

$$\Omega^{\text{top}}_{g,n}(\tilde{\epsilon}_{a_1} \otimes \cdots \otimes \tilde{\epsilon}_{a_n}) = \begin{cases} (\sqrt{\Delta_i})^{2g-2+n} & \text{if } a_1 = \cdots = a_n = i \\ 0 \quad \text{else} \end{cases}$$

Proof. We first compute $\Omega^{\text{top}}_{0,3}$:

$$\Omega^{\text{top}}_{0,3}(\tilde{\epsilon}_{a_1} \otimes \tilde{\epsilon}_{a_2} \otimes \tilde{\epsilon}_{a_3}) = \eta(\tilde{\epsilon}_{a_1} \star \tilde{\epsilon}_{a_2}, \tilde{\epsilon}_{a_3}) = \sqrt{\Delta_{a_1}}\delta_{a_1,a_2}\delta_{a_1,a_3}$$

For general $\Omega^{\text{top}}_{g,n}$, we use the axioms of CohFT. The map r^* contribute a factor $\sqrt{\Delta_i}$:

$$\Omega^{\text{top}}_{g,n}(\tilde{\epsilon} \otimes \cdots \otimes \tilde{\epsilon}) = \Omega^{\text{top}}_{g,n-1}(\tilde{\epsilon} \otimes \cdots \otimes \tilde{\epsilon})\Omega^{\text{top}}_{0,3}(\tilde{\epsilon} \otimes \tilde{\epsilon} \otimes \tilde{\epsilon}) = \sqrt{\Delta}\Omega^{\text{top}}_{g,n-1}(\tilde{\epsilon} \otimes \cdots \otimes \tilde{\epsilon})$$

The map q^* contribute a factor Δ_i each time:

$$\Omega^{\text{top}}_{g,n}(\tilde{\epsilon} \otimes \cdots \otimes \tilde{\epsilon}) = \sum_j \Omega^{\text{top}}_{g-1,n+2}(\tilde{\epsilon} \otimes \cdots \otimes \tilde{\epsilon} \otimes \tilde{\epsilon}_j \otimes \tilde{\epsilon}_j) = \Delta\Omega^{\text{top}}_{g-1,n}(\tilde{\epsilon} \otimes \cdots \otimes \tilde{\epsilon})$$

Repeat this procedure, we get the result. ◻

The free energy $\mathcal{F}^{\Omega^{\text{top}}}_g$ w.r.t $\{\tilde{\epsilon}_i\}$ is

$$\mathcal{F}^{\Omega^{\text{top}}}_g(\mathbf{t}) = \sum_{i=1}^{N}(\sqrt{\Delta_i})^{2g-2+n}\sum_{n,\vec{k}}\frac{1}{n!}\int_{\overline{\mathcal{M}}_{g,n}}\mathbf{t}^i(\psi) \wedge \cdots \wedge \mathbf{t}^i(\psi)$$

The partition function of Ω^{top} w.r.t $\{\tilde{\epsilon}_i\}$ is

$$Z_{\Omega^{\text{top}}}(\hbar; \mathbf{t}) = \exp\sum_g \hbar^{g-1}\mathcal{F}^{\Omega^{\text{top}}}_g(\mathbf{t}) = \prod_{i=1}^{N} Z_{\text{KdV}}(\hbar\Delta_i; \sqrt{\Delta_i}\mathbf{t}^i)$$

This is the reason why some materials, etc [8], define the $\widehat{\Delta}$-action as:

$$\widehat{\Delta}\prod_{i=1}^{N} Z(\hbar; \mathbf{t}^i) := \prod_{i=1}^{N} Z(\hbar\Delta_i; \sqrt{\Delta_i}\mathbf{t}^i)$$

Remark 2.18. *[1] uses*

$$\widehat{\Delta}\prod_{i=1}^{N} Z(\hbar; \boldsymbol{t}^i) := \prod_{i=1}^{N} Z(\hbar\sqrt{\Delta_i}; \boldsymbol{t}^i)$$

because they use $\sum_{g,n}\frac{\hbar^{2g-2+n}}{n!}F_{g,n}$ *in the definition of* Z_Ω.

Similar to the reconstruction of ancestor potential, the partition function of Ω is described by $\widehat{R} Z_{\Omega^{\text{top}}}$. Take the dilaton shift into consideration,

$$Z_\Omega(\hbar;\mathbf{t}) = \widehat{R} Z_{\Omega^{\text{top}}}(\hbar;\mathbf{t}) = \left[\exp\left(\frac{\hbar}{2}\sum_{k,l,\alpha,\beta} V_{kl}^{\alpha\beta}\frac{\partial}{\partial q_k^\alpha}\frac{\partial}{\partial q_l^\beta}\right) Z_{\Omega^{\text{top}}}(\hbar;\mathbf{t})\right](R^{-1}\mathbf{t}(z)+T(z)), \tag{1}$$

where

$$T(z) = z(\mathbf{1} - R^{-1}(z)\mathbf{1}) \in z^2 V[[z]], \quad \mathbf{1} \text{ is unit in } V.$$

The Wick expansion expresses this formula into a graph sum. Before change of variables $\bar{\mathbf{t}} = R^{-1}\mathbf{t} + T$,

$$\frac{\hbar}{2} V_{kl}^{\alpha\beta}\frac{\partial}{\partial q_k^\alpha}\frac{\partial}{\partial q_l^\beta} Z_{\Omega^{\text{top}}}(\hbar;\mathbf{t}) = Z_{\Omega^{\text{top}}}(\hbar;\mathbf{t}) \cdot \Bigg(\sum_g \hbar^{g-1} \sum_{\substack{n,\vec{\alpha},\vec{k} \\ \alpha=\alpha_{j_1},k=k_{j_1} \\ \beta=\alpha_{j_2},l=l_{j_2}}}$$

$$\frac{1}{n!}\int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}^{\text{top}}(\phi_{\alpha_1}\otimes\cdots\otimes\phi_{\alpha_n})\frac{\hbar}{2}V_{kl}^{\alpha\beta}\psi_{j_1}^k\psi_{j_2}^l \prod_{\substack{i=1 \\ i\neq j_1,j_2}}^n \psi_i^{k_i} t_{k_i}^{\alpha_i}\Bigg)$$

Equation (1) becomes:

$$Z_\Omega(\hbar;\mathbf{t}) = Z_{\Omega^{\text{top}}}(\hbar;\bar{\mathbf{t}}) \cdot \sum_g \hbar^{g-1} \sum_{n,\vec{\alpha},\vec{k}} \frac{1}{n!}\int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}^{\text{top}}(\phi_{\alpha_1}\otimes\cdots\otimes\phi_{\alpha_n})$$

$$\sum_{l\leq\lfloor n/2\rfloor}\frac{1}{l!}\sum_{\substack{\vec{\gamma},\vec{\sigma},\vec{c},\vec{d} \\ l(\vec{\gamma})=l(\vec{\sigma})=l}} \frac{\hbar^l}{2^l} \prod_{\substack{m=1 \\ \gamma_m=\alpha_{i_m},\sigma_m=\alpha_{j_m} \\ c_m=k_{i_m},d_m=k_{j_m}}}^{l} V_{i_m,j_m}^{\gamma_m,\sigma_m}\psi_{i_m}^{k_{i_m}}\psi_{j_m}^{k_{j_m}} \prod_{\substack{i=1 \\ i\neq i_m,j_m}}^n \psi_i^{k_i}\bar{t}_{k_i}^{\alpha_i}\Bigg|_{\bar{\mathbf{t}}=R^{-1}\mathbf{t}+T}$$

Take $\log$ on both sides and compare the power of $\hbar$

$$\sum_g \hbar^{g-1}\mathcal{F}_g^\Omega(\mathbf{t}) = \sum_{g,n}\sum_{\Gamma\in G_{g,n}} \frac{\hbar^{g-1}}{|\mathrm{Aut}(\Gamma)|}\int_{\prod_v \overline{\mathcal{M}}_{g(v),n(v)}} \prod_v \Omega_{g(v),n(v)}^{\text{top}}(\phi_{\vec{\alpha}(v)})$$

$$\cdot \prod_{e\in E(\Gamma)}\left(\hbar V_{k_1(e),k_2(e)}^{\alpha_1(e),\alpha_2(e)}\psi_{i_1(e)}^{k_1(e)}\psi_{i_2(e)}^{k_2(e)}\right)\cdot\prod_{l\in L}\psi_l^{k(l)}\bar{t}_{k(l)}^{\alpha(l)}\Bigg|_{\bar{\mathbf{t}}=R^{-1}\mathbf{t}+T}$$

Here $\Gamma \in G_{g,n}$ is a decoracted connected graph with n ordinary leaves:

- the vertex v is labelled with genus $g(v)$, number of adjacent half edges $n(v)$.
- the edge e is labelled with height $k_1(e)$, $k_2(e)$, cohomology class $\alpha_1(e)$, $\alpha_2(e)$.
- the leaf l is labelled with height $k(l)$, cohomology class $\alpha(l)$.

Remark 2.19. *The factor $\frac{1}{2^l}$ disappears in the edge contribution by the symmetry of $V_{kl}^{\alpha\beta}$. See Figure 2.*

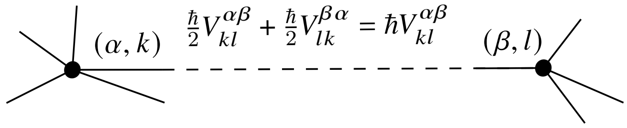

Figure 2: contribution of edges

The change of coordinates is as follows:

$$R^{-1}\mathbf{t} = \sum_{k,l}(R^{-1})_k t_l^{\alpha}\phi_\alpha z^{k+l} = \sum_{k,l}[(R^{-1})_k]^{\alpha}_{\beta} t_l^{\beta}\phi_\alpha z^{k+l}; \quad [R^{-1}\mathbf{t}]_k^{\alpha} = \sum_{0\le l\le k}[(R^{-1})_{k-l}]^{\alpha}_{\beta} t_l^{\beta}$$

About $T(z) = \sum_k T_k z^k$, we know $T_0 = T_1 = 0$ and hence for $k \ge 2$

$$T_k^{\alpha} = [z^{k-1}](-[R^{-1}(z)]^{\alpha}_{\mathbf{1}}) = -[(R^{-1})_k]^{\alpha}_{\mathbf{1}}$$

$$\bar{t}_k^{\alpha} = \sum_{0\le l\le k}[(R^{-1})_{k-l}]^{\alpha}_{\beta} t_l^{\beta} + T_k^{\alpha}$$

$$\sum_g \hbar^{g-1}\mathcal{F}_g^{\Omega}(\mathbf{t}) = \sum_{g,n}\sum_{\Gamma\in G_{g,n}}\frac{\hbar^{g-1}}{|\mathrm{Aut}(\Gamma)|}\int_{\prod_v \overline{\mathcal{M}}_{g(v),n(v)}}\prod_v \Omega^{\mathrm{top}}_{g(v),n(v)}(\phi_{\vec{\alpha}(v)})$$
$$\cdot\prod_{e\in E(\Gamma)}\left(\hbar V^{\alpha_1(e),\alpha_2(e)}_{k_1(e),k_2(e)}\psi_{i_1(e)}^{k_1(e)}\psi_{i_2(e)}^{k_2(e)}\right)$$
$$\cdot\prod_{\substack{i=1\\ \text{leave } l_i}}^{n}\psi_i^{k(l_i)}\left(\sum_{0\le s_i\le k(l_i)}[(R^{-1})_{k(l_i)-s_i}]^{\alpha(l_i)}_{\beta_i} t_{s_i}^{\beta_i} + T^{\alpha(l_i)}_{k(l_i)}\right)$$

$$\frac{1}{n!}\langle\mathbf{t}(\psi),\ldots,\mathbf{t}(\psi)\rangle^{\Omega}_{g,n} = \sum_{m\ge 0}\sum_{\Gamma\in G'_{g,n,m}}\frac{1}{|\mathrm{Aut}(\Gamma)|}\int_{\prod_v \overline{\mathcal{M}}_{g(v),n(v)}}\prod_v \Omega^{\mathrm{top}}_{g(v),n(v)}(\phi_{\vec{\alpha}(v)})$$
$$\cdot\prod_{e\in E(\Gamma)}\left(\hbar V^{\alpha_1(e),\alpha_2(e)}_{k_1(e),k_2(e)}\psi_{i_1(e)}^{k_1(e)}\psi_{i_2(e)}^{k_2(e)}\right)$$
$$\cdot\prod_{i=1}^{n}\psi_i^{k(l_i)}\left(\sum_{k(l_i)\ge s_i}[(R^{-1})_{k(l_i)-s_i}]^{\alpha(l_i)}_{\beta_i} t_{s_i}^{\beta_i}\right)\cdot\prod_{s=1}^{m}T^{\alpha(l_{n+s})}_{k(l_{n+s})}\psi_{n+s}^{k(l_{n+s})}$$

Here $G'_{g,n,m}$ is a connected graph with $g(\Gamma) = g$, n ordinary leaves, and m dilaton leaves. The dilaton leaves is labelled with height $k(l)$ and cohomology class $\alpha(l)$. There are two ways to write this formula as a graph sum.

♠ The first one is originated from [8], also used in [10]. This way emphasizes the formula of the partition function.

- Vertex v: $n(v)$ is the adjacent half edges (including edges, ordinary leaves, and dilaton leaves), k_i is the height of these half edges, α_i is the cohomology classes of these half edges.

$$\mathrm{Cont}(v) = \hbar^{g(v)-1}\int_{\overline{\mathcal{M}}_{g(v),n(v)}}\Omega^{\mathrm{top}}_{g(v),n(v)}(\phi_{\vec{\alpha}(v)})\psi_1^{k_1}\ldots\psi_{n(v)}^{k_{n(v)}}$$

- Edges e:

$$\mathrm{Cont}(e) = [w^{k_1(e)} z^{k_2(e)}] \left(\hbar \frac{\eta^{-1} - R^{-1}(w)\eta^{-1}(R^{-1})^T(z)}{z+w}\right)^{\alpha_1(e)}_{\alpha_2(e)}$$

- Ordinary leaves l:

$$\mathrm{Cont}(l) = [z^{k(l)}](R^{-1}(z)\mathbf{t}(z))^{\alpha(l)}$$

- Dilaton leaves l:

$$\mathrm{Cont}(l) = [z^{k(l)-1}](-[R^{-1}(z)]^{\alpha(l)}_{\mathbf{1}})$$

$$\log Z_\Omega(\hbar;\mathbf{t}) = \sum_{g,n,m} \sum_{\Gamma\in G'_{g,n,m}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{v\in V(\Gamma)} \mathrm{Cont}(v) \prod_{e\in E(\Gamma)} \mathrm{Cont}(e) \prod_{l\in L(\Gamma)\cup L'(\Gamma)} \mathrm{Cont}(l)$$

Remark 2.20. *In [8], the row index is the lower index and the column index is the upper index. The quantization $\widehat{R}$ is defined as $R(-z)^\wedge$ in our notation.*

♣ The another way is from [19], also used in [11, 16, 18]. This way emphasizes the expression of now CohFT. Let

$$T\Omega^{\mathrm{top}}_{g,n}(\phi_{\alpha_1},\ldots,\phi_{\alpha_n}) := \sum_{m\geq 0} \frac{1}{m!}(p_m)_*\Omega^{\mathrm{top}}_{g,n+m}(\phi_{\alpha_1},\ldots,\phi_{\alpha_n},T(\psi_1),\ldots,T(\psi_m))$$

where $p_m : \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$ is the forgetful map. The CohFT $\Omega_{g,n} = \widehat{R}\Omega^{\mathrm{top}}_{g,n}$ is given by

$$\widehat{R}\Omega^{\mathrm{top}}_{g,n}(\phi_{\alpha_1},\ldots,\phi_{\alpha_n}) = RT\Omega^{\mathrm{top}}_{g,n}(\phi_{\alpha_1},\ldots,\phi_{\alpha_n}) = \sum_{\Gamma\in \mathsf{G}_{g,n}} \frac{1}{|\mathrm{Aut}(\Gamma)|}\mathrm{Cont}(\Gamma)$$

Here we emphasize that the label of $\mathsf{G}_{g,n}$ is different from $G_{g,n}$. $\Gamma\in\mathsf{G}_{g,n}$ has its ordinary leaves labelled by $\alpha_1,\ldots,\alpha_n$. There is no label on the edge and no height label on ordinary leaves.

- at each vertex, we place $T\Omega^{\mathrm{top}}_{g(v),n(v)}$
- at each ordinary leave, we place $R^{-1}(\psi_l)\phi_{\alpha(l)}$
- at each edge, we place

$$V(\psi_{i_1(e)},\psi_{i_2(e)}) = \frac{\eta^{-1} - R^{-1}(\psi_{i_1(e)})\eta^{-1}(R^{-1})^T(\psi_{i_2(e)})}{\psi_{i_1(e)}+\psi_{i_2(e)}}$$

$$V(\psi',\psi'') = \sum_{k,l} V^{\alpha,\beta}_{k,l}\phi_\alpha(\psi')^k\otimes\phi_\beta(\psi'')^l \in V\otimes V\otimes H^*(\overline{\mathcal{M}}_{g',n'})\otimes H^*(\overline{\mathcal{M}}_{g'',n''})$$

Cont(Γ) is

$$\prod_{v\in V(\Gamma)} T\Omega^{\text{top}}_{g(v),n(v)}\left(\prod_{e\in E(\Gamma)} V(\psi_{i_1(e)},\psi_{i_2(e)})\otimes\prod_{l\in L(\Gamma)} R^{-1}(\psi_l)\phi_{\alpha(l)}\right)$$

We can also introduce the dilaton leaves (also not labeled) by expanding the vertex contribution $T\Omega^{\text{top}}_{g,n}$. The vertex is replaced by $\Omega^{\text{top}}_{g(v),n(v)}$. The contribution of dilaton sheaves l is $T(\psi_l)$. For such graph $\Gamma\in \mathsf{G}'_{g,n,m}$,

$$\text{Cont}(\Gamma)=\prod_{v\in V(\Gamma)}\Omega_{g(v),n(v)}\left(\prod_{e\in E(\Gamma)} V(\psi_{i_1(e)},\psi_{i_2(e)})\prod_{l\in L(\Gamma)} R^{-1}(\psi_l)\phi_{\alpha(l)}\prod_{s\in L'(\Gamma)} T(\psi_s)\right)$$

The factor $1/m!$ appears in $|\text{Aut}(\Gamma)|$ because m (unlabelled) dilaton leaves connected to the same vertex contribute $m!$ automorphisms.

3 Topological recursion

The topological recursion is a recursion definition of a family of symmetric meromorphic forms $\{\omega_{g,n}\}_{g\in\mathbb{N},n\in\mathbb{N}_+}$ from a spectral curve $\mathcal{S}$.

Definition 3.1. A spectral curve $\mathcal{S}$, is the data of :

$$\mathcal{S}=(C,x,y,B)$$

- ♦ C is a plane complex curve with coordinate (x,y)
- ♦ x and y are two analytical functions $C\to\mathbb{C}$
- ♦ $B(z,z')$ is called the Bergman kernel. It is a symmetric 2nd kind differential on $C\times C$, having a double pole at $z=z'$ and no other pole. It behaves like

  $$B(z,z')\underset{z\to z'}{\sim}\frac{dz\otimes dz'}{(z-z')^2}+O(1)$$

  in any local parameter z.

The point a such that $dx(a)=0$ is called a branch point. Let's restrict to a specific class of $\mathcal{S}$:

Definition 3.2. A spectral curve $\mathcal{S}$ is called regular if:

- the differential form dx has a finite number (non vanishing) of zeros $dx(a_i)=0$, and all zeros of dx are simple zeros.
- The differential form dy does not vanish at the zeros of dx, i.e. $dy(a_i)\neq 0$.

In this case, the spectral curve just has simple branch points. It means that near $x(a_i)$, $x(z) - x(a_i)$ has a double zero, and thus $\zeta(z) = \sqrt{x(z) - x(a_i)}$ is a good local coordinate. Let $\bar{z}$ denote the point corresponding the other sign of $\zeta(z)$, i.e.

$$\zeta(\bar{z}) = -\zeta(z), \quad x(z) = x(\bar{z}).$$

If dy does not vanish it means that

$$y(z) \sim y(a_i) + y'(a_i)\sqrt{x(z) - x(a_i)} + O(x(z) - x(a_i)), \quad y'(a_i) \neq 0.$$

The recursion kernel with z in the vicinity of a branch point a is

$$K_a(z_0, z) = \frac{1}{2} \frac{\int_{z'=\bar{z}}^{z} B(z_0, z')}{(y(z) - y(\bar{z}))dx(z)}$$

$K_a(z_0, z)$ is a meromorphic 1-form in the variable z_0, globally defined on $z_0 \in C$. It has a simple pole at $z_0 = z$ and $z_0 = \bar{z}$. On the other hand, concerning z, $K_a(z_0, z)$ is defined only locally near the branch point a. $K_a(z_0, z)$ is symmetric under involution:

$$K_a(z_0, z) = K_a(z_0, \bar{z})$$

Definition 3.3. The topological recursion is a recursive definition of a family of meromorphic forms $\omega_{g,n}(\mathcal{S}; z_1, \dots, z_n)$ with $g \geq 0$ and $n \geq 1$ via the recursion kernel $K_a(z_0, z)$:

$$\omega_{0,1}(z) = y(z)dx(z)$$

$$\omega_{0,2}(z_1, z_2) = B(z_1, z_2)$$

For $2g - 2 + n > 0$ and $J = \{z_1, \dots, z_n\}$

$$\omega_{g,n+1}(z_0, J) = \sum_{a=\text{branch points}} \operatorname*{Res}_{z \to a} K_a(z_0, z) \Big[\omega_{g-1,n+2}(z, \bar{z}, J) + \sum_{h+h'=g, I \uplus I'=J}' \omega_{h,1+|I|}(z, I)\omega_{h',1+|I'|}(\bar{z}, I')\Big] \tag{2}$$

where $\sum'$ means we exclude the terms $(h, I) = (0, \emptyset), (g, J)$.

The symplectic invariant $\mathcal{F}_g = \omega_{g,0}$ is defined for $g \in \mathbb{N}$. For $g \geq 2$,

$$\mathcal{F}_g = \frac{1}{2 - 2g} \sum_{a=\text{branch points}} \operatorname*{Res}_{z \to a} \omega_{g,1}(z)\Phi(z)$$

where $\Phi(z)$ is any function defined locally near a such that

$$d\Phi = ydx.$$

The following paragraph mainly comes from [11]. Let $c[a]$ be a constant, and $x = (c[a]\zeta_a)^2 + x(a)$ be a local expansion of the ramification point a. Consider the auxiliary function ξ^a and the associated meromorphic forms $d\xi^{k,a}$, defined as

$$\xi^a(z) = \int^z \frac{B(w,-)}{d\zeta_a(w)}\bigg|_{w=a}, \quad d\xi^{k,a}(z) = d\left(\left(-\frac{1}{\zeta_a}\frac{d}{d\zeta_a}\right)^k \xi^a(z)\right)$$

We also set

$$\Delta^a = \frac{dy(z)}{d\zeta_a(z)}\bigg|_{z=a}; \quad t^a = -2c[a]^2 c\Delta^a.$$

Then we see $\{t^a\}$ give a topological CohFT on $V = \mathbb{C}\langle e_1, \dots, e_r\rangle$ by setting $\eta(e_a, e_b) = \delta_{a,b}$ and

$$\mathbf{1} = \sum_{a=\text{branch points}} t^a e_a, \quad \Omega^{\text{top}}_{g,n}(e_{a_1} \otimes \cdots \otimes e_{a_n}) = \frac{\delta_{a_1,\dots,a_n}}{(t^{a_i})^{2g-2+n}}$$

These t^{a_i} play the role of $\Delta_i^{-1/2}$ in previous sections. Define the R-matrix $R \in \mathrm{End}(V)[[u]]$ and the translation $T \in u^2 V[[u]]$ by setting

$$(R^{-1})^b_{\ a}(u) = -\sqrt{\frac{u}{2\pi}} \int_{\gamma_b} d\xi^a e^{-\frac{x-x(b)}{2c[b]^2 u}},$$

$$T^a(u) = \left(t^a u - (-2c[a]^2 c)\sqrt{\frac{u}{2\pi}} \int_{\gamma_a} dy e^{-\frac{x-x(a)}{2c[a]^2 u}}\right)$$

Here $\gamma_a = \{z \in C \mid x(z) - x(a) > 0\}$ is the steepest descent from a, oriented from the negative to the positive values of the local coordinate ζ_a. We define a new CohFT by $\Omega = RT\Omega^{\text{top}}$.

Theorem 3.4 (Eynard-DOSS correspondence)**.** Let a be the set of ramification points.

$$\omega_{g,n}(z_1, \dots, z_n) = c^{2g-2+n} \sum_{\vec{a}\in\mathsf{a}^n, \vec{k}\in\mathbb{N}^n} \langle \tau_{k_1}(e_{a_1}) \dots \tau_{k_n}(e_{a_n})\rangle_g^\Omega \prod_{i=1}^n d\xi^{k_i,a_i}(z_i)$$

where

$$\langle \tau_{k_1}(e_{a_1}) \dots \tau_{k_n}(e_{a_n})\rangle_g^\Omega = \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(e_{a_1} \otimes \cdots \otimes e_{a_n}) \prod_{i=1}^n \psi_i^{k_i}.$$

Moreover, all the ingredients on the RHS depend on the choice of constants $\{c[a]\}_{a\in\mathsf{a}}$ and c, while the LHS is independent of it.

4 R-matrix in Gromov-Witten theory

Let X be a dimension D variety which admits a $\mathbb{C}^*$ equivariant action with isolated fixed points $\mathfrak{p}$'s and finitely many 1-dimensional orbits. The Gromov-Witten CohFT is

$$\Omega^X_{g,n} = \sum_\beta Q^\beta \Omega^X_{g,n,\beta}$$

$$\Omega^X_{g,n,\beta}(\phi_{\alpha_1} \otimes \cdots \otimes \phi_{\alpha_n}) = \pi_*[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}} \cap \prod_{i=1}^n \mathrm{ev}_i^* \phi_{\alpha_i}.$$

The corresponding CohFT is semisimple. We can get the R-matrix in the following step:

- Compute degree 0 Gromov-Witten CohFT. Identify the constant term of R-matrix via quantum Riemann-Roch theorem.
- Multiply fundamental solution S-matrix by the constant term of R-matrix, we get the R-matrix for Gromow-Witten CohFT.

4.1 degree 0 Gromov-Witten CohFT

By $[\overline{\mathcal{M}}_{g,n}(X,0)]^{\mathrm{vir}} = e(T_X \boxtimes \mathbb{E}_g^\vee) \cap [X \times \overline{\mathcal{M}}_{g,n}]$

$$\begin{aligned}
\mathcal{F}^X_g(\mathbf{t})\big|_{Q=0} &= \sum_n \frac{1}{n!} \int_{[\overline{\mathcal{M}}_{g,n}(X,0)]^{\mathrm{vir}}} \prod_{i=1}^n \mathrm{ev}_i^* \phi_{\alpha_i} \bar{\psi}_i^{k_i} t^{\alpha_i}_{k_i} \\
&= \sum_n \frac{1}{n!} \int_{X \times \overline{\mathcal{M}}_{g,n}} e(T_X \boxtimes \mathbb{E}_g^\vee) \prod_{i=1}^n \mathrm{ev}_i^* \phi_{\alpha_i} \bar{\psi}_i^{k_i} t^{\alpha_i}_{k_i} \\
&= \sum_n \frac{1}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \langle e(T_X \boxtimes \mathbb{E}_g^\vee) \prod_{j=1}^n \phi_{\alpha_j}, [X] \rangle \prod_{j=1}^n \psi_j^{k_j} t^{\alpha_j}_{k_j} \\
&= \sum_n \frac{1}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \sum_{\mathfrak{p}} \langle \frac{e(T_{\mathfrak{p}} X \boxtimes \mathbb{E}_g^\vee)}{e(T_{\mathfrak{p}} X)} \prod_{j=1}^n i_{\mathfrak{p}}^* \phi_{\alpha_j}, [\mathfrak{p}] \rangle \prod_{j=1}^n \psi_j^{k_j} t^{\alpha_j}_{k_j}
\end{aligned}$$

Assume the weight of $T_{\mathfrak{p}} X$ is $w_1, \dots, w_r$. (For convenience, we omit the subscript $\mathfrak{p}$.)

$$e(T_{\mathfrak{p}} X) = \prod_{i=1}^D w_i$$

Assume $\mathbb{E}_g = \oplus_{j=1}^g L_j$,

$$\frac{e(T_{\mathfrak{p}} X \boxtimes \mathbb{E}_g^\vee)}{e(T_{\mathfrak{p}} X)} = \frac{\prod_{i,j}(w_i - c_1(L_j))}{\prod_i w_i} = \prod_i \mathbf{c}_{\frac{1}{w_i}}(\mathbb{E}_g^\vee),$$

where $\mathbf{c}_t(\mathbb{E}_g) := \sum_i t^i\lambda_i$, $\mathbf{c}_t(\mathbb{E}_g^\vee) := \sum_i (-1)^i t^i \lambda_i$. Mumford's relation tells that $\mathbf{c}_t(\mathbb{E}_g)\mathbf{c}_t(\mathbb{E}_g^\vee) = 1$,

$$\frac{e(T_\mathfrak{p}X \boxtimes \mathbb{E}_g^\vee)}{e(T_\mathfrak{p}X)} = \prod_i \frac{1}{\mathbf{c}_{\frac{1}{w_i}}(\mathbb{E}_g)}.$$

The deg 0 Gromov-Witten CohFT is

$$\Omega_{g,n}^{X,\beta=0}(\phi_{\alpha_1} \otimes \cdots \otimes \phi_{\alpha_n}) = \sum_\mathfrak{p} \langle \prod_{i=1}^D \frac{1}{\mathbf{c}_{\frac{1}{w_i}}(\mathbb{E}_g)} \prod_{j=1}^n i_\mathfrak{p}^* \phi_{\alpha_j}, [\mathfrak{p}]\rangle$$

We can choose a basis in $\phi_\mathfrak{p} \in H^*_{\mathbb{C}^*}(X) \otimes \mathbb{C}(v)$ such that

$$\phi_\mathfrak{p} = \frac{[\mathfrak{p}]}{e(T_\mathfrak{p}X)}; \quad \phi_\mathfrak{p} \cup \phi_\mathfrak{q} = \delta_{\mathfrak{p},\mathfrak{q}}\phi_\mathfrak{p}; \quad (\phi_\mathfrak{p}, \phi_\mathfrak{q}) = \frac{\delta_{\mathfrak{p},\mathfrak{q}}}{e(T_\mathfrak{p}X)}.$$

Under the basis $\{\phi_\mathfrak{p}\}$

$$\Omega_{g,n}^{X,\beta=0}(\phi_{\mathfrak{p}_1} \otimes \cdots \otimes \phi_{\mathfrak{p}_n}) = \delta_{\mathfrak{p}_1,\ldots,\mathfrak{p}_n} \prod_{i=1}^D \frac{1}{\mathbf{c}_{\frac{1}{w_i}}(\mathbb{E}_g)}.$$

The above statement of degree 0 Gromov-Witten CohFT depends on the choice of coordinates, and we will see it is not suitable to use quantum Riemann-Roch theorem because $\mathbb{E}_g$ is not of the form $\pi_*\mathrm{ev}_{n+1}^*E$ for any vector bundle over a point $\mathfrak{p}$. There is another statement of degree 0 Gromov-Witten CohFT which overcomes these drawbacks.

$$\begin{aligned}\mathcal{F}_g^X(\mathbf{t})\big|_{Q=0} &= \sum_n \frac{1}{n!} \int_{[\overline{\mathcal{M}}_{g,n}(X,0)]^{\mathrm{vir}}} \prod_{i=1}^n \mathrm{ev}_i^*\phi_{\alpha_i}\bar{\psi}_i^{k_i} t_{k_i}^{\alpha_i} \\ &= \sum_n \frac{1}{n!} \sum_\mathfrak{p} \int_{\overline{\mathcal{M}}_{g,n}(\mathfrak{p})} \frac{\prod_{i=1}^n \mathrm{ev}_i^*\phi_{\alpha_i}\bar{\psi}_i^{k_i} t_{k_i}^{\alpha_i}}{e_{\mathbb{C}^*}(R^\bullet\pi_*\mathrm{ev}_{n+1}^*T_\mathfrak{p}X)}\end{aligned}$$

where $R^\bullet\pi_*\mathrm{ev}_{n+1}^*T_\mathfrak{p}X = \oplus_{i=1}^D F_{g,n}^i$

$$e_{\mathbb{C}^*}(R^\bullet\pi_*\mathrm{ev}_{n+1}^*T_\mathfrak{p}X) = \prod_{i=1}^D w_i^{1-g}\mathbf{c}_{\frac{1}{w_i}}(F_{g,n}^i).$$

If we adapt the classical normalized canonical basis $\tilde{\phi}_\mathfrak{p}$, $\mathbf{t} = t_k^l\tilde{\phi}_{\mathfrak{p}_l}z^k$, then

$$\mathcal{F}_g^X(\mathbf{t})\big|_{Q=0} = \sum_n \frac{1}{n!} \sum_{\mathfrak{p}_l} \int_{\overline{\mathcal{M}}_{g,n}} \frac{(\prod_{i=1}^D w_{i,l})^{g-1+\frac{n}{2}} \prod_{j=1}^n \psi_j^{k_j} t_{k_j}^l}{\prod_{i=1}^D \mathbf{c}_{\frac{1}{w_i}}(F_{g,n}^i)}$$

$$\mathcal{D}_X(\hbar;\mathbf{t})|_{Q=0} = \prod_{\mathfrak{p}_l} \mathcal{D}_{\mathrm{pt}}^{\mathrm{tw}}(\hbar e(T_{\mathfrak{p}_l}X); \mathbf{t}^l\sqrt{e(T_{\mathfrak{p}_l}X)}),$$

where at each $\mathfrak{p}$,

$$\langle \tau_{k_1}\tilde{\phi}_{\alpha_1}, \ldots, \tau_{k_n}\tilde{\phi}_{\alpha_n}\rangle_g^{\mathrm{tw}} = \int_{\overline{\mathcal{M}}_{g,n}} \frac{\prod_{i=1}^n \mathrm{ev}_i^*\tilde{\phi}_{\alpha_i}\psi_i^{k_i}}{\prod_{i=1}^D \mathbf{c}_{\frac{1}{w_i}}(F_{g,n}^i)}$$

4.2 quantum Riemann-Roch theorem

This section is from [3, 4, 21]. Consider the universal family over $X_{g,n,\beta} = \overline{\mathcal{M}}_{g,n}(X,\beta)$.

$$\begin{array}{ccc} X_{g,n+1,\beta} & \xrightarrow{ev_{n+1}} & X \\ \pi \downarrow & & \\ X_{g,n,\beta} & & \end{array}$$

Given a holomorphic vector bundle E over X, set

$$E_{g,n,\beta} = \pi_* \mathrm{ev}_{n+1}^* E \in K^0(X_{g,n,\beta})$$

Given an invertible multiplicative characteristic class of complex vector bundles $\mathbf{c}(\cdot)$, define the $(\mathbf{c}, E)$-twisted genus g Gromov-Witten potential to be

$$\mathcal{D}_{\mathbf{c},E} = \exp \sum_g \hbar^{g-1} \mathcal{F}^g_{\mathbf{c},E}$$

$$\mathcal{F}^g_{\mathbf{c},E}(\mathbf{t}) = \sum_{n,\beta} \frac{Q^\beta}{n!} \langle \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi); \mathbf{c}(E_{g,n,\beta})\rangle_{g,n,\beta}$$

$$\langle \tau_{k_1}(v_1), \ldots, \tau_{k_n}(v_n); \mathbf{c}(E_{g,n,\beta})\rangle_{g,n,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \wedge_{i=1}^n \mathrm{ev}_i^* v_i \psi_i^{k_i} \wedge \mathbf{c}(E_{g,n,\beta})$$

Assume the invertible multiplicative characteristic class $\mathbf{c}(\cdot)$ takes the form

$$\mathbf{c}(\cdot) = \exp \sum_{k \geq 0} s_k \mathrm{ch}_k(\cdot),$$

we write $\mathbf{s} = (s_0, s_1, s_2, \ldots)$ throughout, and use $\mathcal{F}^g_{\mathbf{s}}$ to indicate $\mathcal{F}^g_{\mathbf{c},E}$. The quantum Riemann-Roch theorem tells

Theorem 4.1 ([3, 4]).

$$\exp\left(-\frac{1}{24}\sum_{l>0} s_{l-1} \int_X \mathrm{ch}_l(E) c_{D-1}(T_X)\right)(\mathrm{s}\det\sqrt{\mathbf{c}(E)})^{-\frac{1}{24}} \mathcal{D}_{\mathbf{s}} =$$
$$\exp\left(\sum_{m>0}\sum_{l\geq 0} s_{2m-1+l} \frac{B_{2m}}{(2m)!}(\mathrm{ch}_l(E) z^{2m-1})^\wedge\right) \exp\left(\sum_{l>0} s_{l-1}(\mathrm{ch}_l(E)/z)^\wedge\right)\mathcal{D}_X$$

When the target X in Theorem 4.1 becomes the $\mathbb{C}^*$-fixed points $\mathfrak{p}$ in Section 4.1, the twisted characteristic class is given by $E_{g,n,\beta} = \pi_* \mathrm{ev}_{n+1}^* T_{\mathfrak{p}} X$,

$$\mathbf{c}(E_{g,n,\beta}) = \frac{1}{\prod_i \mathbf{c}_{\frac{1}{w_i}}(F^i_{g,n})} = \frac{1}{\prod_{i,j} \mathbf{c}_{\frac{1}{w_i}}(x_j)} = \frac{1}{\prod_{i,j}(1 + x_j/w_i)}$$

By Taylor expansion,

$$\frac{1}{1+\frac{x_j}{w_i}} = \exp\sum_{k\geq 1} s^i_k \frac{x_j^k}{k!}, \quad s^i_k = (k-1)!(-w_i)^{-k}$$

$$\mathbf{c}(E_{g,n,\beta}) = \exp\sum_{i=1}^{D}\sum_{k\geq 1} s^i_k \mathrm{ch}_k(E_{g,n,\beta}) = \exp\sum_{k\geq 1} s_k \mathrm{ch}_k(E_{g,n,\beta})$$

where $s_k = \sum_{i=1}^{D}(-1)^k s^i_k = (k-1)!\sum_i(-w_i)^{-k}$. Because $X = \mathfrak{p}$, the vector bundle E over $\mathfrak{p}$ is trivial,

$$\mathrm{ch}_l(E) = \begin{cases} 1, & l = 0 \\ 0, & l \geq 1 \end{cases}$$

In other words, under the classical normalized canonical basis $\{\phi_{\mathfrak{p}}\}$, $\mathbf{t} = t^l_k \tilde{\phi}_{\mathfrak{p}_l} z^k$, the quantum Riemann-Roch theorem tells that

$$\mathcal{D}_X(\hbar;\mathbf{t})|_{Q=0} = \prod_{\mathfrak{p}_l} \mathcal{D}^{\mathrm{tw}}(\hbar e(T_{\mathfrak{p}_l}X);\mathbf{t}^l\sqrt{e(T_{\mathfrak{p}_l}X)})$$

$$= \prod_{\mathfrak{p}_l} \exp\left(\sum_{m\geq 1} -\frac{B_{2m}}{2m(2m-1)}\left(\sum_{i=1}^{D} w_{i,l}^{-2m+1}\right) z^{2m-1}\right)^{\wedge} \mathcal{D}_{\mathrm{pt}}(\hbar e(T_{\mathfrak{p}_l}X);\mathbf{t}^l\sqrt{e(T_{\mathfrak{p}_l}X)})$$

$$= \prod_{\mathfrak{p}_l} \exp\left(\sum_{m\geq 1} -\frac{B_{2m}}{2m(2m-1)}\left(\sum_{i=1}^{D} w_{i,l}^{-2m+1}\right) z^{2m-1}\right)^{\wedge} \mathcal{D}_{\Omega^{\mathfrak{p}_l}}(\hbar;\mathbf{t}^l)$$

Here $\Omega^{\mathfrak{p}}$ is a 1-dimensional topological CohFT with cup product rescaled by the Euler class $e(T_{\mathfrak{p}}X) = \prod_i w_i$:

$$\tilde{\phi}_{\mathfrak{p}} \cup \tilde{\phi}_{\mathfrak{p}} = \sqrt{\prod w_i}\,\tilde{\phi}_{\mathfrak{p}}; \quad \eta(\tilde{\phi}_{\mathfrak{p}},\tilde{\phi}_{\mathfrak{p}}) = 1.$$

Then we know the R-matrix on Ω^X have the property that $R(z)|_{Q=0} = \exp\mathrm{diag}(b_1,\ldots,b_N)$, where

$$b_l(z) = \sum_{m\geq 1}\left(\sum_{i=1}^{D}(-w_{i,l})^{-2m+1}\right)\frac{B_{2m}}{2m}\frac{z^{2m-1}}{2m-1}.$$

Let $t = t' + t'' \in H^*(X)$, where $t' \in H^2(X)$. When we use this theorem in Gromov-Witten theory, we identify $Q := Qe^{\langle t',\beta\rangle}$. Then $Q = 0$ is equivalent to say $t' \to -\infty$. The constant ambiguity of R-matrix is fixed by

$$\lim_{\substack{t'\to-\infty \\ t''\to 0}} R(z,t)^j_{\ l} = \delta^j_l \sum_{m\geq 1}\left(\sum_{i=1}^{D}(-w_{i,l})^{-2m+1}\right)\frac{B_{2m}}{2m}\frac{z^{2m-1}}{2m-1},$$

where $t'' \to 0$ means we go back to classical normalized canonical basis. It recovers [14, Theorem 9.1].

Notations	this note	[14]	[15]
fixed points	$\mathfrak{p}_l$	w_i	w_i
weights	$w_{i,l}$ tangent weight	$\chi_j(w_i)$ cotangent weight	χ^i_r tangent weight
$\dim X$	D	n	
	$\sum_{i=1}^{D}(-w_{i,l})^{-2m+1}$	$N^{(i)}_{2k-1} = \sum_{j=1}^{n} \chi_j^{-2k+1}(w_i)$	$-N_{2k-1}(1/\chi^i) = -\sum_{r=1}^{n}(\chi^i_r)^{-2k+1}$

Let $R^i_j(z,u)$ be the R-matrix of ${S^i}_j = \delta^i_j + \ll \frac{\phi_j}{z-\psi}, \phi^i \gg_{0,2}(\tau)$ w.r.t $S = Re^{\mathbf{u}/z}$ such that $u^i(0) = 0$. Then

$$ {}^{\text{new}}{R^j}_l := {R^j}_l \sum_{m\geq 1}\left(\sum_{i=1}^{D}(-w_{i,l})^{-2m+1}\right)\frac{B_{2m}}{2m}\frac{z^{2m-1}}{2m-1} $$

is the R-matrix of Gromov-Witten CohFT $\Omega^X_{g,n}$.

A Linear Algebra

There are various notations of linear algebra from different authors. It caused some confusion for me when I want to compare papers from different origins.

I get used to use M^a_b to denote $Me_b = M^a_b e_a$. a is row index, b is column index. As far as I know, [3, 17, 19] use this notation. In this note, I use this notation by default.

$$ M(e_1,\ldots,e_n) = (e_1,\ldots,e_n)(M^a_b). $$

$$ M_{ab} = (e_a, Me_b), \quad M^a_b = (e^a, Me_b) $$

The adjoint of M is

$$ \langle Mu, v\rangle = \langle u, M^* v\rangle. $$

$$ (M^*)^a_b = (e^a, M^* e_b) = (Me^a, e_b) = {M_b}^a. $$

The transpose $(M^T)^a_b = M^b_a = \eta^{bc}{M_c}^d\eta_{da}$. Hence, $M^T = \eta M^* \eta^{-1}$.

On the other hand, [8, 14, 15] uses low index b as a row, and upper index a is a column. There are two conventions. One way is to use right multiplication (in [8]). On this setting, $e_b M = {N_b}^a e_a$.

$$ \begin{pmatrix} e_1 \\ \cdots \\ e_n \end{pmatrix} M = {N_b}^a \begin{pmatrix} e_1 \\ \cdots \\ e_n \end{pmatrix} $$

$$ (N\circ M)e_a = (e_a M)N = {M_a}^b e_b N = {M_a}^b {N_b}^c e_c \longleftrightarrow NMe_a = N^c_b M^b_a e_c $$

$$ (Nt)^a = t^b {N_b}^a, \quad {(NM)_a}^b = {M_a}^c {N_c}^b. $$

In this notation, $N_b^{\ a} = M^a_{\ b} = (M^T)^b_{\ a}$. i.e. N is the transpose of M. In this case, the upper index means coefficient of the vector field:

$$t = t^a e_a = (e_1, \dots e_n) \begin{pmatrix} t^1 \\ \dots \\ t^n \end{pmatrix} = (t^1, \dots, t^n) \begin{pmatrix} e_1 \\ \dots \\ e_n \end{pmatrix}$$

$$Mt = (e_1, \dots e_n)[M^a_{\ b}] \begin{pmatrix} t^1 \\ \dots \\ t^n \end{pmatrix} = (t^1, \dots, t^n)[N_a^{\ b}] \begin{pmatrix} e_1 \\ \dots \\ e_n \end{pmatrix}$$

The another "row down column up" notation $M_a^{\ b}$ is to use metric η to change the indices (in [15]). $M_a^{\ b} := \eta_{ac} M^c_{\ d} \eta^{db}$, $e^i = \eta^{ij} e_j$

$$Mt = (e^1, \dots e^n)[M_a^{\ b}] \begin{pmatrix} t_1 \\ \dots \\ t_n \end{pmatrix}$$

$$(N \circ M)_a^{\ b} = N_a^{\ c} M_c^{\ b}$$

The lower index is the coefficient of the (co)vector field.

Remark A.1. *In [15, Equation 7], the paper says expanding the first row of $S(z)$, it means the lower index is row (use the η to change index):*

$$S_{\mathbf{1}}^{\ i}(z) = \langle \mathbf{1}, S(\phi^i) \rangle = \langle \sum \delta^\mu \phi_\mu, S(\phi^i) \rangle = \sum \delta^\mu S_\mu^{\ i}$$

$$S_{\mathbf{1}}^{\ i}(z) = \langle \sum \Delta_j^{-1/2} (\Delta_j^{1/2} \partial_j), S_k^{\ i} \phi^k \rangle = \langle \sum \Delta_j^{-1/2} (\Delta_j^{1/2} \partial_j), R_k^{\ i} e^{u^i/z} \phi^k \rangle = \sum_j \Delta_j^{-1/2} R_j^{\ i} e^{u^i/z}$$

In the text below [15, Equation 24], $S^i_\mu = e^{u^i/z} R^i_j \Psi^j_\mu$ means

$$S_\mu^{\ i} = \Psi_\mu^{\ j} R_j^{\ i} e^{u^i/z}.$$

B Dubrovin's notations

This section aims to compare the notations in [6, 7, 12]. It is not a complete introduction of Dubrovin's theory. In [6, 7], u_i's are canonical coordinates,

$$\psi_{i1} := \sqrt{\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i} \rangle}$$

The transition matrix $\Psi = (\psi_{i\alpha}(u))$ is given by

$$\frac{\partial}{\partial t^\alpha} = \sum_{i=1}^n \frac{\psi_{i\alpha}}{\psi_{i1}} \frac{\partial}{\partial u_i}$$

The rotation coefficients $\gamma_{ij}, (i \neq j)$ is

$$\gamma_{ij}(u) := \frac{\partial_j \sqrt{\eta_{ii}(u)}}{\sqrt{\eta_{jj}(u)}} = \frac{1}{2} \frac{\partial_i \partial_j t_1(u)}{\sqrt{\partial_i t_1(u) \partial_j t_1(u)}}$$

Proposition B.1. The rotation coefficients γ_{ij} is $(R_1)^i{}_j$ in the fundamental solution.

Proof. Recall the formula $\Psi d\Psi^{-1} = [d\mathbf{u}, R_1]$. The matrix $\Psi d\Psi^{-1}$ is given by the Christoffel symbols. In the canonical coordinates, $\Gamma^i{}_j = \Gamma^i_{kj} du^k$, where

$$\Gamma^k_{ij} = \frac{1}{2}\Delta_k \left(\delta_{jk} \frac{\partial(\Delta_k^{-1})}{\partial u_i} + \delta_{ik} \frac{\partial(\Delta_i^{-1})}{\partial u_j} - \delta_{ij} \frac{\partial(\Delta_i^{-1})}{\partial u_k} \right)$$

The transition between $\tilde{\epsilon}_i$ and ϵ_i ($\epsilon_i = \sqrt{\Delta_i}^{-1}\tilde{\epsilon}_i$) tells that under the normalized canonical basis $\{\tilde{\epsilon}_i\}$.

$$\nabla = d + \sqrt{\Delta}^{-1} d\sqrt{\Delta} + \sqrt{\Delta}^{-1}\Gamma\sqrt{\Delta},$$

$$\Psi d\Psi^{-1} = \sqrt{\Delta}^{-1} d\sqrt{\Delta} + \sqrt{\Delta}^{-1}\Gamma\sqrt{\Delta}$$

Then we get

$$[\Psi d\Psi^{-1}]^i{}_j = \frac{1}{2}\sqrt{\Delta_i}\sqrt{\Delta_j}\left(\frac{\partial(\Delta_i^{-1})}{\partial u^j} du^i - \frac{\partial(\Delta_j^{-1})}{\partial u^i} du^j \right).$$

Expand $[d\mathbf{u}, R_1]^i{}_j = (R_1)^i{}_j(du^i - du^j)$, we know when $i \neq j$,

$$(R_1)^i{}_j = \frac{1}{2}\sqrt{\Delta_i}\sqrt{\Delta_j}\frac{\partial(\Delta_i^{-1})}{\partial u^j} = \gamma_{ij}.$$ □

Remark B.2. *I think [17, page 19] makes a mistake on the power of $\sqrt{\Delta}$.*

Remark B.3. *The tablet summarizes the notations in [6, 7, 12]:*

Givental	Dubrovin
$\Delta_i^{-1/2}$	ψ_{i1}
$\frac{1}{\Delta_i}$	$\eta_{ii} = \psi_{i1}^2 = \partial_i t_1$
$\Psi^i{}_\alpha$	$\psi_{i\alpha}$
$(R_1)^i{}_j$	γ_{ij}

By this identification, we can deduce some relations about $\psi_{i\alpha}, \gamma_{ij}$ in [6, 7].

Lemma B.4. *For distinct i, j, k,*

$$\partial_k \gamma_{ij} = \gamma_{ik}\gamma_{kj}, \quad k \neq i, j$$

Proof. The flatness condition of ∇ tells

$$d(\Psi d\Psi^{-1}) + \Psi d\Psi^{-1} \wedge \Psi d\Psi^{-1} = 0$$

By $[\Psi d\Psi^{-1}]^i{}_j = \gamma_{ij}(du^i - du^j)$,

$$\partial_k \gamma_{ij} du^k \wedge (du^i - du^j) + \gamma_{ik}(du^i - du^k) \wedge \gamma_{kj}(du^k - du^j) = 0$$

Then we get the result. □

Lemma B.5.

$$\partial_k \psi_{i\alpha} = \gamma_{ik} \psi_{k\alpha}, \quad k \neq i.$$

Proof. $-d\Psi \cdot \Psi^{-1} = \Psi d\Psi^{-1} = [d\mathbf{u}, R_1]$, so

$$d\Psi = [R_1, d\mathbf{u}]\Psi, \quad d\psi_{i\alpha} = \gamma_{ik}(du^k - du^i)\psi_{k\alpha}. \qquad \square$$

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